Fermi gas response to time-dependent perturbations

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We describe the Riemann-Hilbert (RH) approach to computing the long-time response of a Fermi gas to a time-dependent perturbation. The approach maps the problem onto a non-commuting RH problem. The method is non-perturbative, quite general and can be used to compute the Fermi gas response in driven (out of equilibrium) as well as equilibrium systems. It has the appealing feature of working directly with scattering amplitudes defined at the Fermi surface rather than with the bare Hamiltonian. We illustrate the power of the method by rederiving standard results for the core-hole and open-line Greens functions for the equilibrium Fermi edge singularity (FES) problem. We then show that the case of the non-separable potential can be solved non-perturbatively with no more effort than for the separable case. We compute the corresponding results for a biased (non-equilibrium) model tunneling device, similar to those used in single photon detectors, in which a photon absorption process can significantly change the conductance of the barrier. For times much larger than the inverse bias across the device, the response of the Fermi gases in the two electrodes shows that the equilibrium Fermi edge singularity is smoothed, shifted in frequency and becomes polarity-dependent. These results have a simple interpretation in terms of known results for the equilibrium case but with (in general complex-valued) combinations of elements of the scattering matrix replacing the equilibrium phase shifts. We also consider the shot noise spectrum of a tunnel junction subject to a time-dependent bias and demonstrate that the calculation is essentially the same as for the FES problem. For the case of a periodically driven device we show that the noise spectrum for the Coherent States of Alternating Current found in [1] can be easily obtained using this approach.

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I. INTRODUCTION

An approach to the study of the quantum statistics of an **arbitrary** single-particle observable in a Fermi gas has been recently described in [2]. We refer to it as the RH approach, as it reduces the calculation of a determinant describing the quantum statistics of an observable to the solution of an (in general) non-Abelian Riemann-Hilbert (RH) problem. The relation between such determinants and RH problems has been known for a long time, and has been used extensively in studies of quantum inverse scattering problems [3]. As a result, a lot is known about the non-abelian RH problem [4], and much of this can be taken over directly to the study of the quantum statistics of Fermi gases.

The RH approach gives an expression for the distribution function of an observable in a Fermi gas perturbed by a time-dependent potential. To illustrate the method, one of us used it to prove a long-standing conjecture, first stated in [5], that the two sources of shot noise in a biased point contact, namely fluctuations in the number of attempts to tunnel through the barrier and fluctuations in the number of reflections, are statistically independent [2]. We have also used the method to study how non-equilibrium effects alter the Fermi Edge Singularity in a tunnel junction [6].

The response of a Fermi gas to a time-dependent perturbing potential is a central problem in condensed matter physics. It has been tackled in many different contexts often with different approaches. For systems out of equilibrium, such as quantum pumps, perturbative approaches, based on the Keldysh formalism, have been used, while for systems in equilibrium it has been possible to find exact solutions in some limiting cases by solving the equations of motion directly [7, 8, 9, 10]. One of the advantages of the RH method is that it applies equally to all such problems and therefore offers the prospect of a unified approach to computing the time-dependent response of all observables in Fermi gases.

Setting up the description of a problem in the RH framework is quite straightforward. Given the solution of the single-particle scattering problem, the response of the Fermi gas reduces to the computation of a determinant of an operator taken over single-particle states occupied in the initial configuration. (The generalization of the method to the more general case in which the initial state is given in terms of a density matrix rather than a single quantum state should be possible but has not yet been formulated.) The evaluation of this determinant then reduces to the solution of a Riemann Hilbert problem. The solution is in general a matrix-valued function analytic everywhere except across a cut, along which the function is discontinuous. The discontinuity is fixed by the driving force or perturbation acting upon the system [2]. From the point of view of the Keldysh formalism the method performs a non-trivial re-summation of all relevant diagrams with the help of the solution of the corresponding RH problem. In the abelian case, when the discontinuity function commutes with itself at all points along the cut, the solution is given in terms of an integral. The

classic solution of the FES problem [8, 9] is the simplest example of this solution. In the non-abelian case, the solution to the RH problem is not known in general, although asymptotic solutions exist. These are valid for response frequencies small compared to those present in the discontinuity function.

Here we explain the RH approach in some detail. To illustrate the power of the method we start by showing how the solution of the equilibrium FES problem [8, 9, 11] is derived. We then show the generalization of this problem to include the case where the 'impurity' potential mixes scattering states of the unperturbed problem—the case of a non-separable potential—and deal explicitly with the case when the impurity potential gives rise to a bound state. This problem was treated initially in [10, 12], in a calculation which solved directly the Dyson-like equation for the appropriate Green's functions. In the RH formulation of this problem the discontinuity function, although matrix-valued, is constant and commutes with itself. As a consequence, the solution to the RH problem is trivial to derive and yields the standard results of [10, 12] with no more work than for the case separable potential case. We show how these results are changed in a non-equilibrium situation. In both the equilibrium and non-equilibrium cases, we compute both the core-hole Green's function reported in [6] and the open line contribution. Finally we show how the states which minimize the shot noise in a periodically driven quantum pump—the so-called Coherent States of Alternating Current (CSAC's) [1]—can be described using the RH method.

II. PERTURBING THE FERMI GAS

We consider a system in which particles impinge upon a localized potential (see Fig 1). The potential is time-independent for all times $t < t_0$ and $t > t_f$. For times $t_0 < t < t_f$ the potential varies. We take our basis to be the eigen states of the system with the localised potential at its value. The states are labeled by their single-particle energy ϵ ($\hbar = 1$) and a channel index i. We will consider the corresponding annihilation operator, $a_{i\epsilon}$, as the i'th component of the vector $\hat{\mathbf{a}}_\epsilon$. The Hamiltonian of the system is then

$$\begin{aligned} \hat{H}(t) &= \hat{H}_0 + \sum_{\epsilon,\epsilon'} \hat{\mathbf{a}}^+_\epsilon M(t,\epsilon,\epsilon')\hat{\mathbf{a}}_{\epsilon'} \\ \hat{H}_0 &= \sum_\epsilon \epsilon \hat{\mathbf{a}}^+_\epsilon \hat{\mathbf{a}}_\epsilon. \end{aligned} \tag{1}$$

Here $M(t,\epsilon,\epsilon') = 0$ when $t < 0$ or $t > t_f$. (In the following, for any operator $\hat{O}$, we will denote by O the matrix of $\hat{O}$ taken between the single-particle basis states.)

We will be interested in the total effect of the perturbation, *ie* what is the final state of the system for $t > t_f$ given the initial state at $t = 0$. This requires a knowledge of the effect on the initial many-body state of the time-development operator $\hat{U}(t_f)$, where

$$i\frac{d\hat{U}}{dt} = \hat{H}(t)\hat{U}(t), \quad \hat{U}(0) = 1. \tag{2}$$

Because the Hamiltonian $\hat{H}(t)$ in (1) is quadratic, the effect of $\hat{U}(t_f)$ is fully characterized by its effect on the set of **single-particle** scattering states, $a^+_{i\epsilon'}|\rangle$:

$$\hat{U}(t_f)\hat{\mathbf{a}}^+_{\epsilon'}|\rangle = \sum_\epsilon e^{-i\epsilon t_f}\sigma(\epsilon,\epsilon')\hat{\mathbf{a}}^+_\epsilon|\rangle, \tag{3}$$

where $\sigma(\epsilon,\epsilon')$ is some unitary $N \times N$ and $|\rangle$ is the true vacuum with no particles in the system [13].

When computing the response of the Fermi gas to the time-dependent potential, we will need to compute expectation values of the type:

$$\chi_R = \langle 0|\hat{R}|0\rangle. \tag{4}$$

Here $|0\rangle$ is the state of the Fermi gas before the perturbation is applied and $\hat{R}$ is an operator (or operator product) related to an obvervable of interest. In general, the $\hat{R}$ in (4) will involve the time-development operator $\hat{U}(t_f)$. For example, in the case of the shot noise spectrum of a tunneling barrier [2], the interest is in the statistics of the charged transferred from one electrode to the other. If $\hat{Q}_1$ is the charge in the first electrode then the expectation value of $\hat{R} = \hat{U}^\dagger(t_f)e^{i\lambda\hat{Q}_1}\hat{U}(t_f)e^{-i\lambda\hat{Q}_1}$, yields the generating function for moments of the distribution of charge transferred out of channel 1 (into channel 2) during the period between $t = 0$ and $t = t_f$. In the case of the FES problem [6], the core-hole Green's function (see below) is related to the overlap $\langle 0|\hat{U}^\dagger_0(t_f)\hat{U}(t_f)|0\rangle$ where $\hat{U}_0(t_f)$ is the time-development operator for $\hat{H}_0$ in (1).

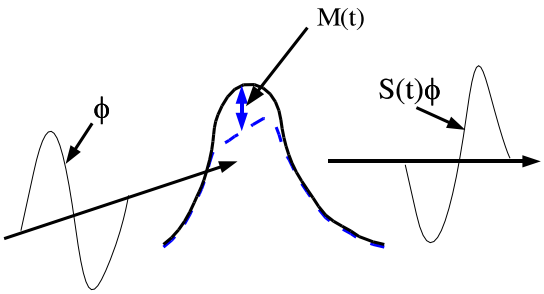

FIG. 1: Schematic illustration of the generic scattering problem. Particles impinge on a localized potential which is independent of time for times $t < t_0$ and $t > t_f$. For times $t_0 < t < t_f$ the potential varies as a function of time. The instantaneous value of the potential at time t is shown as the dashed curve and the difference measured with respect to the time-independent potential is denoted by $M(t)$. When the potential varies slowly (condition 5), the effect of the scattering potential on an incident partial wave (shown schematically as ϕ) is simply multiplication by the scattering matrix corresponding to the instantaneous value of $M(t)$.

This overlap is an expectation value of the type (4) with $\hat{R} = \hat{U}_0^\dagger(t_f)\hat{U}(t_f)$.

The effect of $\hat{U}(t_f)$ acting on the single-particle states of the basis is given by the unitary matrix $\sigma(\epsilon, \epsilon')$ defined in (3). The matrix $\sigma(\epsilon, \epsilon')$ can be related to the scattering matrix, $S(t, E)$, for a particle with energy E evaluated on the *instantaneous* value of the potential $M(t)$ in (1). This reflects the fact that $S(t, E)$ encodes all the information in the potential variations $M(t)$. This relation will be complicated in general. However, when

$$\hbar \frac{\partial S^{-1}}{\partial t} \frac{\partial S}{\partial E} \ll 1, \tag{5}$$

the relation between σ and S becomes simple:

$$\begin{aligned} \sigma_{l\epsilon l'\epsilon'} &= S_{l\epsilon l'\epsilon'} \\ S_{l\epsilon l'\epsilon'} &= \frac{1}{2\pi\sqrt{\nu_l \nu_{l'}}} \int dt S_{ll'}(t, E) e^{i(\epsilon-\epsilon')t}, \end{aligned} \tag{6}$$

where $E = (\epsilon + \epsilon')/2$ and ν_i is the density of states in channel i. The result (6) shows that the total scattering amplitude from state k in channel n to k' in channel n' is just the Fourier transform of the scattering matrix $S(t)$ evaluated on the instantaneous value of the potential. This result is well-known. It was used implicitly to solve the FES problem in the presence of a separable potential [11] and is often used to simplify studies of quantum pumps.

A brief derivation of the condition (5) is given in [2]. The condition can be understood heuristically as follows (see also [1]). We consider the incoming wave-packet to be a partial wave in channel n of the basis, in which S is diagonal before the perturbation is switched on. After impinging on the potential, the partial waves scattered from channel n into channel n' will take a time of order of the corresponding Wigner delay time to pass out of the region where the potential acts. The condition (5) is equivalent to the requirement that the scattering matrix does not change significantly during this delay time. If this condition is satisfied, the relation (6) also has a simple interpretation. A wave with energy $\epsilon' = k'v$ in channel n' incident at time t on the potential (which is assumed to be localized around the origin) will have amplitude at the origin proportional to $e^{-ik'vt}$ where v is the velocity of the incident wave. Waves will emanate from the source at the origin with amplitude $S_{nn'}(t)e^{-ik'vt}$ in channel n. If dispersion effects are small this will lead to a waveform $S_{nn'}(t-(r/v'))e^{-ik'v'(t-(x/v'))}$. Decomposing this into waves of the form $e^{ik(x-v't)}$ as $t \to t_\infty$ gives the result (6). The normalization factor $1/\sqrt{\nu_l \nu_l'}$ is included so that in the case where the incoming flux ($\sim v|\psi_n|^2$) is totally scattered into channel n' the scattering amplitude is 1.

A. Fermi Sea at $T = 0$

The calculation of the response of the Fermi gas to the time-dependent perturbation reduces to the computation of the expectation value $\chi_R = \langle 0|\hat{R}|0\rangle$ in (4). In the following we will assume that $|0\rangle$ is a single Slater determinant. In this case the computation of χ_R requires the evaluation of a single determinant:

$$\langle 0|\hat{R}|0\rangle = \det{}' \mid R \mid, \tag{7}$$

where the elements of R are given by:

$$R_{ii'}(\epsilon, \epsilon') = \langle |\hat{a}_{i\epsilon} \hat{R} \hat{a}^\dagger_{i'\epsilon'}| \rangle. \tag{8}$$

and where the prime on the det indicates that the determinant is taken only over states occupied in $|0\rangle$.

When the initial Slater determinant $|0\rangle$ corresponds to a filled Fermi sea, it is useful to introduce the Fermi distribution as an operator with elements:

$$f_{\epsilon,\epsilon'} = \delta_{\epsilon,\epsilon'} \delta_{ii'} \theta(-(\epsilon - \mu)). \tag{9}$$

Here μ is the chemical potential, which we take to be zero. (For the non-equilibrium problems discussed later the chemical potential can vary according to the channel index. However, a time-dependent change of basis replaces the differing chemical potentials in the different electrodes by an additional time-dependence in the scattering matrix $S(t)$, so that there is no loss of generality in assuming $\mu = 0$.) In a block notation that separates the states with positive and negative energies f and R become:

$$f = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad R = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}. \tag{10}$$

and

$$1 - f + fR = \begin{pmatrix} R_{11} & R_{12} \\ 0 & 1 \end{pmatrix}. \tag{11}$$

It then follows that

$$\chi_R = \det{}' \mid R_{11} \mid = \det|1 - f + fR| \tag{12}$$

where det is now a determinant taken over all states in the basis.

Expressing χ_R as the determinant of $1 - f + fR$ taken over all states in the basis, allows us to write

$$\log \chi_R = \mathrm{Tr}\,(f \ln R) + \mathrm{Tr}\,(\ln(1 - f + fR) - f \ln R) \tag{13}$$

In (13), we have added and subtracted the term $\log \chi_R^{(1)} \equiv \mathrm{Tr}\,(f \ln R)$. This term consists of the diagonal elements of $\ln R$ summed over all occupied states in $|0\rangle$ and gives the contributions linear in t_f. It often has a simple physical interpretation. In the FES problem it yields the threshold shift (or change in the ground state energy of the Fermi gas after the core hole is created), while in the shot noise spectrum of the tunneling barrier it can be shown to be related to the average transfer of charge across the barrier (the Brouwer formula [2, 14]).

The second term in (13),

$$\log \chi_R^{(2)} \equiv \mathrm{Tr}\,(\ln(1 - f + fR) - f \ln R), \tag{14}$$

accounts for all the non-trivial effects associated with excitations close to the Fermi surface induced by the perturbation. (States far from the Fermi energy, when $f = 1$ or $f = 0$, make no contribution to this term. As a result there are no problems associated with effects of the band edge or short time cutoff when computing this term.) In the case of the FES problem it describes the line-shape, while in the shot noise spectrum it gives all the higher moments of the charge transfer distribution. When computing this term, we will later switch to the time-representation in which

$$R_{l\epsilon l'\epsilon'} = \frac{1}{2\pi\sqrt{\nu_l\nu_{l'}}}\int dt R_{ll'}(t,E)e^{i(\epsilon-\epsilon')t}, \qquad (15)$$

with (as in 6) $E = (\epsilon+\epsilon')/2$. R will normally involve σ or some simple combination of σ with itself and its inverse. Provided the condition (5) is satisfied, we will be able to evaluate σ by ignoring the dependence of $S(t,E)$ on E. (This E dependence is not important as states far from the Fermi energy do not contribute to $\log\chi_R^{(2)}$.) As a result the term $\log\chi_R^{(2)}$ will depend only on the time-dependence of the scattering matrix evaluated at the Fermi energy, which we will denote by $S(t)$.

B. The Riemann-Hilbert Problem

Computing the second term in (13), $\log\chi_R^{(2)}$, is the central task in the evaluation of the response of the Fermi gas. The non-trivial part of this is finding the inverse of $(1-f+fR)$ which can then be used in an integral representation for its logarithm. This inverse can be written in terms of the solution of an $N\times N$ matrix Riemann-Hilbert problem, where N is the length of the vector $\hat{\mathbf{a}}_\epsilon$, *ie* the number of channels in the problem (see 1).

A standard procedure for representing the logarithm of an infinite matrix, such as the one on the right hand side of (14), introduces a λ-dependence for R via [15]

$$R(\lambda) = \exp(\lambda\log R) \qquad (16)$$

and then uses an integral over λ to represent the logarithm:

$$\log\chi_R^{(2)} = \int_0^1 d\lambda \mathrm{Tr}\left[\left((1-f+fR)^{-1}f - fR^{-1}\right)\frac{dR}{d\lambda}\right]. \qquad (17)$$

(The λ-dependence of R introduced in (16) is assumed in (17) although not written explicitly.)

To compute the trace in (17), we switch to a time representation in which a quantity A becomes:

$$A_{ll'}(t,t') = \frac{1}{2\pi\sqrt{\nu_l\nu_{l'}}}\int_{-\infty}^{\infty}\nu_l d\epsilon\int_{-\infty}^{\infty}\nu_{l'}d\epsilon' A_{ll'}(\epsilon,\epsilon') \qquad (18)$$

Now, the Fermi distribution (9) is no longer diagonal:

$$f_{ll'}(t,t') = \frac{i}{2\pi}\frac{\delta_{ll'}}{t-t'+i0}. \qquad (19)$$

However, as we can neglect the E dependence of $R(\lambda,t,E)$ and $S(t,E)$ (see discussion after 15), R and S are now diagonal in t and equal to $R(\lambda,t,0)\delta(t-t')$ and $S(t,0)\delta(t-t')$ respectively. In the time-representation, the product of two quantities requires matrix multiplication in the space of scattering channels together with an integral over the intermediate time coordinate. Where one of the quantities in the product is diagonal in t (for example S), the integral over the intermediate time coordinate is of course trivial and the product reduces to the simple matrix multiplication in the channel space. The Tr now becomes a trace over the scattering channels, which we denote by tr, and an integral over the time coordinate, so that

$$\log\chi_R^{(2)} = \int_0^1 d\lambda\int dt\,\mathrm{tr}\left[\left((1-f+fR)^{-1}f - fR^{-1}\right)\frac{dR}{d\lambda}\right]. \qquad (20)$$

Here, when computing the diagonal (equal time) elements of O, one should take $\lim_{t\to t'}O(t,t')$. If A and B are diagonal in the time representation, it follows that

$$\begin{aligned}\int dt\,\mathrm{tr}\,[A,f]\,B &= \int dt\,\lim_{t'\to t}\mathrm{tr}\frac{i}{2\pi}\left[\frac{A(t)-A(t')}{t-t'+i0}\right]B(t')\\ &= \frac{i}{2\pi}\int dt\,\mathrm{tr}\,\frac{dA(t)}{dt}B(t), \qquad (21)\end{aligned}$$

which is a result we use later.

The quantity $(1-f+fR)^{-1}$ can be written in terms of the function $Y(t)$, which is a matrix in the channel space and which solves an auxiliary Riemann-Hilbert problem. $Y(t)$ should be analytic everywhere in the complex $t-$plane except on the interval $[0,t_f]$ on the real axis along which it satisfies

$$Y_-(t)Y_+^{-1}(t) = R(t) \qquad \text{where} \qquad Y_\pm = Y(t\pm i0). \qquad (22)$$

In addition Y should satisfy

$$Y\to 1 \qquad \text{when} \qquad |t|\to\infty. \qquad (23)$$

These analytic properties together with (19) yield the useful identities

$$\begin{aligned}fY_-f &= fY_-\\ fY_+f &= Y_+f \qquad (24)\end{aligned}$$

Using these relations (and assuming that Y^{-1} is also analytic everywhere except along the cut), it is then easy to verify that

$$(1-f+fR)^{-1} = Y_+\left((1-f)Y_+^{-1} + fY_-^{-1}\right). \qquad (25)$$

As an aside, we note that $(1-f+fR)^{-1}$ is the solution to a singular integral equation, with f playing the role of the singular kernel of the Cauchy type. It is well known that such integral equations can be solved using Carleman's method, which writes the solution in terms of an analytic function satisfying a Riemann-Hilbert problem

[16]. In the one-channel the corresponding singular integral equation for the case when $R(t)$ is constant between $t = 0$ and $t = t_f$ is the problem solved in [8, 9] when describing the equilibrium FES.

Inserting (25) into (20) and using (21) we obtain

$$\log \chi_R^{(2)} = \frac{i}{2\pi} \int_0^1 d\lambda \int dt \, \mathrm{tr} \left\{ \frac{dY_+}{dt} Y_+^{-1} R^{-1} \frac{dR}{d\lambda} \right\}. \quad (26)$$

The integral over t is over all times. However, as $dR/d\lambda$ normally vanishes for $t > t_f$ and $t < 0$, one often only needs to integrate from 0 to t_f.

Equations (26) and (13) map the characterisation of the response χ_R in (4) onto an integral involving the solution, $Y(t)$, of a Riemann-Hilbert (RH) problem (22 and 23). An appealing feature of this formulation is that these formulas apply for any choice of variable χ_R provided that the Fermi gas (or gases) is initially in its ground state and apply for many non-equilibrium cases as well. If $R(t)$ commutes with itself at different values of t, the solution of the RH problem can be written in closed form. Although there is no solution for the general case, a lot is known about such non-commuting problems including some asymptotic solutions valid when t_f^{-1} is much smaller than any characteristic frequency in $R(t)$ [4].

III. FERMI EDGE SINGULARITY

In this section we show how all the known results for the equilibrium Fermi Edge Singularity (FES) follow directly from the formula (26). Within our formalism the case of non-separable channels considered in [10] and again in [12] is no more complicated than the separable case. We will then discuss how these results are changed in the nonequilibrium case.

A. Equilibrium

The FES problem was first considered in the context of the X-ray absorption spectrum of a metal [7]. When a photon creates a core hole in a metal, the Fermi gas is affected by the potential of the core hole leading to the excitation of particle-hole pairs. The absorption line expected in the absence of the Fermi gas becomes a threshold with a singularity in the absorption spectrum as a function of $\omega - \omega_0 > 0$:

$$I(\omega) \sim \mid \omega - \omega_0 \mid^{-\alpha}, \quad (27)$$

where ω_0 is the threshold frequency for absorption. It turns out that similar singularities are seen in the distribution of energy absorbed by the Fermi gas in response to any rapid change in potential and not just in x-ray absorption experiments. For example, the consequences of the FES are also seen in a tunnel junction. As the energy absorbed by the Fermi gas when switching, is an important characteristic of the device, establishing how the FES changes in such tunneling devices is important for understanding fluctuations in energy transfer across such devices. The FES is also thought to be related to the apparent absence of detailed balance in Random Telegraph Signals [17].

The Hamiltonian for the photon absorption experiment is [8]

$$\hat{H} = \epsilon_0 \hat{b}^\dagger \hat{b} + \sum_\epsilon \epsilon \hat{\mathbf{a}}_\epsilon^+ \hat{\mathbf{a}}_\epsilon + \sum_{\epsilon' \epsilon} \hat{\mathbf{a}}_\epsilon^\dagger V(\epsilon, \epsilon') \hat{\mathbf{a}}_{\epsilon'} \, \hat{b} \hat{b}^\dagger + \hat{H}_X \quad (28)$$

with the operators $\hat{\mathbf{a}}$ as in (1) and $V(\epsilon, \epsilon')$ an $N \times N$ matrix. The operator $\hat{b}^\dagger$ is the creation operator corresponding to the core state and the coupling to the X-ray field is described semiclassically by

$$\begin{aligned} \hat{H}_X &= \sum_\epsilon \mathbf{W}_\epsilon \cdot \hat{\mathbf{a}}_\epsilon^\dagger \hat{b} e^{i\omega t} + h.c. \\ &\equiv \hat{X} e^{i\omega t} \end{aligned} \quad (29)$$

The absorption spectrum is proportional to the real part of the Fourier transform of the response function

$$S(t_f) = \langle 0 | T\{\hat{X}(t_f) \hat{X}(0)\} | 0 \rangle, \quad (30)$$

with T the time-ordering operator. $S(t_f)$ can be computed from the core-hole Green's function [8]

$$G(t_f) = \langle 0 | T\{\hat{b}^\dagger(t_f) \hat{b}(0)\} | 0 \rangle \quad (31)$$

and the function

$$F(t_f) = \sum_{\epsilon, \epsilon'} \langle 0 | T\{\hat{b}^\dagger(t_f) (\mathbf{W}_{\epsilon'}^* \cdot \hat{\mathbf{a}}_{\epsilon'}(t_f))\} (\mathbf{W}_\epsilon \cdot \hat{\mathbf{a}}_\epsilon^\dagger(0)) \hat{b}(0) | 0 \rangle. \quad (32)$$

Coventionally a minus sign is included in the definition of F and G. However, as we will only deal with the absorption case here and take $t_f > 0$, it is easier to work from these definition. We have also left out the conventional factor of i in the definitions of these Green's functions as in [8].

The calculation of F and G reduces to a one-body scattering problem [8, 18]. As far as the Fermi gas is concerned the role of the core hole is to switch on the scattering potential $V(\epsilon, \epsilon')$ at time 0 and switch it off again at t_f. As such, the problem is clearly in the form of (1) with $M(t, \epsilon, \epsilon') = 0$ for $t > t_f$ and $t < 0$ and $M(t, \epsilon, \epsilon') = V(\epsilon, \epsilon')$ for $0 < t < t_f$. The corresponding scattering matrix $S(t, E)$ switches between the identity when the core-hole is absent and some constant value $S^e(E)$ when it is present. The asymptotic behavior of the response at large t_f (when $\omega - \omega_0 \ll \xi_0^{-1}$) is determined by states with energies close to the Fermi surface. For these states we assume that the variation of $S^e(E)$ with E can be neglected so that the condition (5) is satisfied. (The limit $\xi_0 t_f \gg 1$ is the one considered in [8].)

The calculation of G is one of the simplest calculations within the RH approach. G is the expectation value of the operator $\hat{R}$ in (4) with:

$$\hat{R} = \hat{U}_0^\dagger(t_f) \hat{U}(t_f). \quad (33)$$

As the matrix elements of $\hat{U}(t_f)$ are just $e^{-i\epsilon t_f}\sigma_{\epsilon\epsilon'}$, it follows from (7) that [11]

$$G(t_f) = e^{i\epsilon_0 t_f}\det{}' \mid \sigma \mid \tag{34}$$

while from (3, 15 and 16)

$$R(t) = S(t) \quad \text{and} \quad R(\lambda, t) = \exp\lambda \log S(t). \tag{35}$$

The RH problem (22 and 23) reduces to

$$Y_-(t)Y_+^{-1}(t) = S^\lambda(t) \qquad Y \to 1 \text{ when } |t| \to \infty. \tag{36}$$

When the matrix S is constant between 0 and t_f, we will denote its value by S^e. In the single-channel case, $S^e = e^{2i\delta}$ and the RH problem is solved by [16]

$$Y(z) = \exp\left[\frac{1}{2\pi i}\ln\left(\frac{z}{z-t_f}\right)\lambda \log S^e\right]. \tag{37}$$

(This solution was used implicitly in the original solution to the single channel problem of [8].) In fact the result (37) solves the RH problem even where the problem is not separable provided that the matrices $S(t)$ evaluated at different times t with $0 < t < t_f$ commute. (This can be checked by direct substitution into (36).) We insert $Y(z)$ and $R(\lambda, t)$ into (26). The integral over t runs between 0 and t_f where $\log R(\lambda)$ is non-zero. Inserting into (13) and including the factor of $e^{i\epsilon_0 t_f}$ yields

$$\log\chi_R = i\epsilon_0' t - \log i\xi_0 t_f \left(\frac{\delta}{\pi}\right)^2, \tag{38}$$

where $\epsilon_0' = \epsilon_0 + \sum_{\epsilon<0}\delta(\epsilon)/\pi\nu(\epsilon)$, with $\nu(\epsilon)$ the density of states, is the shifted energy of the core-hole in the presence of the Fermi gas. (The form for the difference between ϵ_0 and ϵ_0' is usually attributed to Fumi [19, 20].) Close to the branch points of Y at 0 and t_f, we cut the integrals off at $i\xi_0^{-1}$ and $t_f + i\xi_0^{-1}$ where ξ_0 is an energy of order the band width. Eq (38) gives the well-known result for the long-time asymptotic behavior of G [7]:

$$G(t_f) \sim (i\xi_0 t)^{-\alpha} e^{i\epsilon_0' t_f}, \qquad \alpha = (\delta/\pi)^2. \tag{39}$$

To compute the function $F(t_f)$ in (32) is slightly more involved, although the underlying RH problem is the same. As already mentioned, the role of the core hole is to switch on the potential at $t = 0$ and switch it off again at t_f, so that $F(t_f)$ can be written (writing out the channel indices explicitly)

$$F(t_f) = \sum_{i\epsilon, i'\epsilon'} W_{i\epsilon}^* \langle 0|\hat{r}(i\epsilon, i'\epsilon')|0\rangle W_{i'\epsilon'} \tag{40}$$

$$\hat{r}(i\epsilon, i'\epsilon') = \hat{U}_0^\dagger(t_f)\hat{a}_{i\epsilon}\hat{U}(t_f)\hat{a}_{i'\epsilon'}^\dagger. \tag{41}$$

In the basis of the scattering states $\hat{a}_{j'\alpha'}^+|\rangle$ the matrix elements of this operator are easily shown to be given in terms of σ in (3) by

$$r(i\epsilon, i'\epsilon')_{j\alpha j'\alpha'} = e^{i(\epsilon_0-\epsilon)t_f}\left(\sigma_{j\alpha j'\alpha'}\sigma_{i\epsilon i'\epsilon'} - \sigma_{j\alpha i'\epsilon'}\sigma_{i\epsilon j'\alpha'}\right). \tag{42}$$

Using (7) and (40) we find

$$F(t_f) = e^{i\epsilon_0 t_f}\det{}' \mid C\sigma - |h\rangle\langle g| \mid. \tag{43}$$

Here $C = C(t_f)$ is the number:

$$C = \sum_{i\epsilon, i'\epsilon'} e^{-i\epsilon t_f} W_{i\epsilon}^* \sigma_{i\epsilon i'\epsilon'} W_{i'\epsilon'} \tag{44}$$

while

$$\begin{aligned} |h\rangle &= \sum_{j\alpha}\left(\sum_{i'\epsilon'}\sigma_{j\alpha i'\epsilon'}W_{i'\epsilon'}\right)a_{j\alpha}^\dagger\,|\rangle \\ \langle g| &= \sum_{j'\alpha'}\left(\sum_{i\epsilon} e^{-i\epsilon t_f}W_{i\epsilon}^*\sigma_{i\epsilon j'\alpha'}\right)\langle|a_{j'\alpha'}. \end{aligned} \tag{45}$$

The expression (43) is now in the form (7). We could attempt to solve the corresponding RH problem (22) and (23) as before, although the relation between the corresponding operator $\hat{R}(t)$ and S is no longer simple. However, it is easier to simplify (46) by factoring out $G(t_f) = e^{i\epsilon_0 t_f}\det{}'|\sigma|$:

$$F(t_f) = CG(t_f)\,\det\left|\,1 - C^{-1}O|h\rangle\langle g|\,\right| \tag{46}$$

with

$$O = (1 - f + f\sigma)^{-1}f. \tag{47}$$

We have used (12) to put (46) in the form of the determinant over all states in the basis. Using the identity $\det|1 - C^{-1}O|h\rangle\langle g|| = 1 - C^{-1}\langle g|O|h\rangle$ we obtain

$$F(t_f) = G(t_f)\left(C - \langle g|O|h\rangle\right). \tag{48}$$

As $C \to 0$ for large t_f (with a functional form which depends on assumptions about the density of states at the band edge), the response is determined by the second term.

The function $Y(z)$ with $\lambda = 1$ in (36 and 25) can be used to compute $F(t_f)$. In the time-representation

$$(1 - f + f\sigma)^{-1}f = Y_+ f Y_-^{-1}. \tag{49}$$

Writing $F(t_f) = L(t_f)G(t_f)$ (L is usually referred to as the open-line contribution), we find

$$\begin{aligned} L = -\sum_{ll'}\int d\epsilon d\epsilon' \int dt_1 dt_2 W_{l\epsilon}^* e^{i\epsilon(t_1-t_f)} \times \\ \sqrt{\nu_l}\left[SY_+fY_-^{-1}S\right]_{lt_1,l't_2}\sqrt{\nu_{l'}}e^{-i\epsilon' t_2}W_{l'\epsilon'} \end{aligned} \tag{50}$$

Taking W_i to independent of ϵ (we are assuming that the long t_f behavior is determined by states with energies within $\sim 1/t_f$ of the Fermi surface), this simplifies to give

$$L \simeq W_i\sqrt{\nu_i}\left[Y_-(t_f^-)\frac{1}{it_f}Y_+^{-1}(0^-)\right]_{ii'}\sqrt{\nu_{i'}}W_{i'}. \tag{51}$$

The functions Y_- and Y_+ are evaluated at $t = 0^-$ and $t = t_f^-$. This prescription is equivalent to the imaginary time cutoff used to derive (38) and used in [8, 12]. Strictly, the discontinuities in S at $t = 0$ and $t = t_f$ should be thought of as the limit of a fast switching process, in which S starts to change at $t = 0$ and reaches its new value S^e after a short time. Similarly at $t = t_f$, S starts to change back from S^e to its unperturbed value. (The corrections associated with a more realistic model of a non-instantaneous switching process were considered for a related problem in [21].) In the single channel case we can insert the explicit form for Y given by (37), and recover the standard results

$$L \sim \frac{1}{it_f}\frac{1}{(i\xi_0 t_f)^{-2\delta/\pi}},\ F \sim \frac{1}{it_f}\frac{1}{(i\xi_0 t_f)^{(\delta/\pi)^2-2\delta/\pi}}. \quad (52)$$

When the potential $M(t)$ in (1) is strong enough for a bound state of the Fermi gas electrons to form below the bottom of the band, the results for $G(t_f)$ and $F(t_f)$ are no longer correct. Provided the perturbing potential is constant between 0 and t_f, the effect of the bound state can be taken into account explicitly as explained in the appendix. The result for $G(t_f)$ given by (A2), and for $F(t_f)$ given by (A12), have two main contributions. After taking the Fourier transform to obtain the absorption spectrum, the first corresponds to having the bound state occupied and leads to the absolute threshold for absorption. The second term relates to scattering processes in which the bound state is always empty and leads to a subsidiary threshold at E_b above the first in the absorption spectrum.

The results (39), (52), (A13) are of course very well-known [8, 9]. However, none of the key formulas (37, 51 and A12) require that the scattering matrix S should be diagonal in the channel indices. Provided that $S(t)$ commutes with itself at different times, the results are valid for arbitrary channel number. We can therefore use the function $Y(z)$ given by (37) to compute the corresponding results for the case of a non-separable potential just as easily as in the separable case. In the absence of bound states, one obtains with $\epsilon_0' = \epsilon_0 + \sum_{\epsilon<0}\sum_\zeta \delta_\zeta(\epsilon)/\pi\nu_\zeta(\epsilon)$:

$$\begin{aligned} G(t_f) &= \exp\left(i\epsilon_0' t_f\right)(i\xi_0 t_f)^{-\beta} \\ L(t_f) &= \sum_\zeta |\tau_\zeta|^2 \frac{1}{it_f}\exp\left(\frac{2}{\pi}\delta_\zeta \ln i\xi_0 t_f\right) \\ \beta &= \sum_\zeta \left(\frac{\delta_\zeta}{\pi}\right)^2, \end{aligned} \quad (53)$$

which are the results obtained perturbatively in [12]. Here the eigen values of the matrix S^e (see after 36) are written as $e^{2i\delta_\zeta}$. S^e has eigen vectors $f(\zeta)_i$ and $\tau_\zeta = \sum_i \sqrt{\nu_i} W_i^* f(\zeta)_i$.

The perturbing potential, characterised by scattering matrix S^e, can be strong enough to lead to a bound state with wavefunction given by (A4). In the presence of a bound state(s) we take the eigenvalues of S^e to be $e^{i2\tilde\delta_\zeta}$ with $\tilde\delta_\zeta$ defined as the phase shift modulo π in channel ζ on the interval $[-\pi/2, \pi/2]$ (see discussion after A7). We then obtain the generalizations to the non-separable case of the results of [11] for $G(t_f)$ and $F(t_f)$. We find

$$G(t_f) = \tilde G(t_f)\,(1 + A_B)\,, \quad (54)$$

where $\tilde G(t_f)$ is the contribution of the scattering states given by the expression for G in (53) with phase shifts given by $\tilde\delta_\zeta$, while

$$A_B \sim e^{-iE_B t_f}\sum_\zeta |\eta_\zeta|^2 e^{-2i\tilde\delta_\zeta}\exp\left(-\frac{2}{\pi}\tilde\delta_\zeta \ln i\xi_0 t_f\right). \quad (55)$$

Here $\eta_\zeta = \sum_i \sqrt{\nu_i} u_i^* f(\zeta)_i$ and the u_i are the bound state wavefunction coefficients given in (A4). In the presence of the bound state, the function $F(t_f) \sim F_0(t_f) + F_b(t_f)$ with

$$\begin{aligned} F_0(t_f) &\sim \tilde G(t_f)\tilde L(t_f) \\ F_b(t_f) &\sim e^{-iE_B t_f}\tilde G(t_f)|\mathbf{u}\cdot\mathbf{W}|^2, \end{aligned} \quad (56)$$

where $\tilde L(t_f)$ is the scattering state contribution to $L(t_f)$ given by the expression in (53), using the phase shifts $\tilde\delta_\zeta$.

The formulas (56) and (53) are the natural generalizations of the single channel result and have exactly the same interpretation as was given originally in [9, 22]. We repeat this briefly here as the results for the non-equilibrium case (given in the next section) can also be understood heuristically on a similar basis but with the phase shifts becoming complex. The exponents $(\delta_\zeta/\pi)^2$ and $(\delta_\zeta/\pi \pm 1)^2$ are, according to the Friedel sum rule, the square of the net charge that needs to move in to or away from the origin in order to screen the core hole potential. For $G(t_f)$ this is δ_ζ/π, while for $F(t_f)$ it is $(\delta_\zeta - \pi)/\pi$ if the photoelectron inserted at the origin is in the ζ channel and δ_ζ/π otherwise. If there is an occupied bound state after absorption of the photon, the respective values become $(\tilde\delta_\zeta + \pi)/\pi$ and $\tilde\delta_\zeta/\pi$, as now the Fermi gas has to provide the additional electron which ends up in the bound state. The form t^{-n^2} is just the decay with time of the overlap of the wavefunction of the Fermi gas at $t = t_f$ and the one describing the system created at $t = 0$ in which (with respect to the ground state in the presence of the core hole) there is an excess charge $n = -\delta/\pi$ at the origin. That it vanishes as $t \to \infty$, is the orthogonality catastrophe described by Anderson [23].

B. Non-equilibrium Effects

The experimental and technological interest in the out-of-equilibrium response of coupled Fermi systems has grown as electronic devices have shrunk. Examples include structured quantum dots, like the single electron transistor or the single photon detector [24], and quantum point contacts. The non-equilibrium Fermi Edge

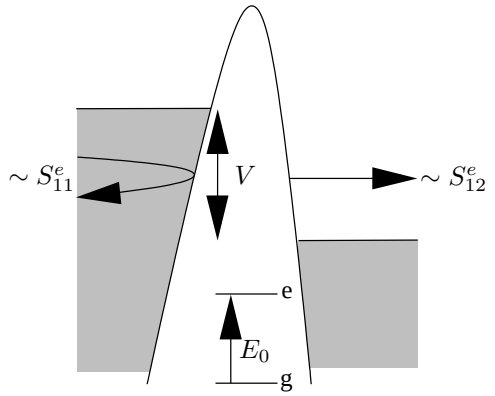

FIG. 2: Energy levels in an idealized device to demonstrate the out-of-equilibrium FES. The scattering potential for electrons is characterized via the 2×2 matrix, $S(\epsilon)$, connecting scattering states in the two wires for particles with energy ϵ. $S = S^g$ or S^e depending on whether the defect is in its ground (g) or excited (e) state (with excitation energy E_0). S^g is the identity matrix and S^e is an arbitrary unitary matrix. S^e_{11} and S^e_{12} correspond to the reflection and transmission amplitudes respectively. We will refer to the device operating as illustrated here, with a negative potential $-V$ ($V > 0$) applied to the left electrode, as the forward-biased case.

Singularity (nFES) will characterize the energy absorbed by the coupled Fermi gases in a rapid switching process in such devices. The nFES should help explain, for example, measurements of random telegraph signals (RTS). In these experiments, a two-level system (TLS) couples to the source-drain current flowing in the channel of a MOSFET (the TLS resides in the insulating oxide layer [17]). The RTS relates to the 'random' switching of the TLS between its ground and excited states. The ratio between the times the TLS spends in the excited and ground states is measured experimentally. In equilibrium this ratio is fixed by detailed balance, and the deviations from this have been attributed to non-equilibrium effects [25].

As one of the simplest non-trivial many-body effects, the FES is also a natural point to start, when looking for a description of non-equilibrium effects in many-electron systems. Perhaps surprisingly, given its conceptual simplicity, the nFES has not attracted as much attention as more difficult non-equilibrium problems like the Kondo effect, to which it is known to be related. (The Kondo effect can be thought of as a sequence of FES's associated with each flipping of the localized moment [26].)

In [6] we reported results for $G(t_f)$ for a two-channel problem, which modelled a system with two electrodes separated by a barrier. The transmission of the barrier depends on the state of a two-level system inside the barrier, see Figure 2, with the transition between the two-levels assumed to be dipolar. The real part of the Fourier transform of the function $G(t_f)$ gives the absorption spectrum for the device. The non-equilibrium effects predicted in [6] should be visible in the voltage dependence of the absorption line-shape of devices like the single-photon detector of [24] and [27]. In [24] a quantum dot in the quantum Hall regime is coupled via tunneling barriers to two electrodes on either side of the dot. For magnetic fields in the range 3.4-4.2T, the conductance through the dot can change from zero to around $0.3\ e^2/h$ when a photon is absorbed via cyclotron resonance in the dot. From the perspective of the two electrodes, the dot behaves as a tunneling barrier which allows tunneling only in its excited state. The absorption of the photon and the subsequent separation of the hole (which moves into the $\nu = 1$ ring on the outer part of the dot) and the particle (which 'falls' into center of the dot at $\nu = 2$) is rapid, while the response of the conduction electrons in the two electrodes is slow and will show effects characteristic of the FES. In the device of [27], a electron trapped in a dot underneath an electron channel gives rise to a potential which closes off a conducting channel. When a photon is absorbed, the photo-exicted hole can recombine with the electron in the trap, the potential of the electron disappears and the channel opens. Again the conduction electrons on the two sides of the channel, 'see' the sudden reduction of a tunneling barrier on absorption of a photon.

The main result reported in [6] was that the ND formula describing the form of the FES and threshold shift (Fumi's theorem [19, 20]) generalized in a simple way to the non-equilibrium case. For time scales $t \ll t_f$, the phase shifts which appear in $G(t_f)$ are real and are given by the log of the eigen-values of the scattering matrix S^e [10, 12]. This simply reflects the fact that on these short time scales the response of the Fermi gas involves excitations with energies $\epsilon \gg V$ which do not sense the non-equilibrium distribution function. On time scales $t \gg t_f$, the equilibrium phase shifts in the two channels are replaced by 'complex' phase shifts given by $\log S^e_{11}$ and $\log (1/S^e_{22})^*$. The real part of these phase shifts describes the scattering within each electrode, while the imaginary part describes the effect of scattering processes in which particles cross the barrier. One effect of the non-equilibrium operation of the device is to make the scattering between the different electrodes effectively incoherent. Here, we find that this interpretation extends also for the function $F(t_f)$.

We show the key steps in the derivation of $G(t_f)$, emphasizing the relationship with the equilibrium results, and report the results for $F(t_f)$ including the role of possible bound states. Since the initial state involves a filled Fermi sea in both channels (left and right electrodes), the RH formulation of this non-equilibrium problem is the same as for the equilibrium case. The bias across the tunnel junction means only that the chemical potentials are different in the two electrodes. One way of handling this difference is to introduce a gauge transformation acting only on the basis states in the left electrode:

$$\mathbf{a}(\epsilon) \ \rightarrow \ \mathbf{a}(\epsilon, t) = e^{+iL\int_0^t V(\tau)d\tau}\mathbf{a}(\epsilon) \tag{57}$$

$$S(t) \ \rightarrow \ S(t) = e^{+iL\int_0^t V(\tau)d\tau}S(t)e^{-iL\int_0^t V(\tau)d\tau} \tag{58}$$

where L is the diagonal matrix with $L_{11} = 1$ and $L_{22} = 0$. The effect of this transformation on states in the left electrode is to set $\epsilon \to \epsilon - V(t)$, so that the chemical potential in the left electrode becomes equal to that in the right electrode (taken to be zero as before). For the constant bias case, the transformation gives $\hat{\mathbf{a}}(t) \to \hat{\mathbf{a}}(t) = e^{+iLVt}\hat{\mathbf{a}}$.

The functions $\log \chi_R^{(2)}$ and $L(t_f)$ for the nFES case are still given by (26) and (50). However, the RH problem satisfied by the function $Y(z)$ is different: In (36) $S^\lambda(t)$ picks up an additional time-dependence from the gauge transformation (58), which leads to two important differences to the equilibrium case. Firstly, the function e^{iVt} introduces a new characteristic energy scale, V. If the function $S(t)$ has Fourier components with freqencies $\omega \gg V$, the response will be dominated by states with energies $|\epsilon| \gg V$ and will be insensitive to the non-equilibrium nature of the distribution which only becomes apparent on the energy scale V. If $S(t)$ only varies at frequencies $\omega \ll V$ the response will come from states with energies $\epsilon \ll V$ and will normally be significantly different from what happens in equilibrium.

The second main difference following from the additional time-dependence of $S(t)$ relates to the case when between $t = 0$ and $t = t_f$ the scattering matrix (before the gauge transformation) is constant and equal to S^e. In this case it is now no longer possible to solve the RH problem with a function of the form (37). Although this form satisfies formally the jump condition, $Y_-(t)Y_+^{-1}(t) = S^\lambda(t)$, the corresponding function $Y(z)$ is not well-defined for large z if S^e is not diagonal. The off-diagonal elements of S^e contain factors $e^{\pm iVt}$ so that there is an essential singularity at $z \to \infty$ in $Y(z)$ defined by (37), and it no longer satisfies the condition $Y \to 1$. This problem is clearly apparent in the RH formulation we have presented. It was much less clear in previous attempts to extend the ND method to the non-equilibrium case and may explain why these failed [28]. It is also interesting to note that for $t_f \ll 1/V$, we can expand the function e^{iVt} up to linear order in Vt_f. Then $S^e = S^e(V = 0) + CVt$, there is no singularity at infinity for Y, and the form (37) still works.

In general there is no exact solution to the non-commuting RH problem [4]. However, in the case relevant to the device shown in Fig 2, $S(t) = e^{iLVt}S^e e^{-iLVt}$ for $0 < t < t_f$, with S^e constant, we can find an asymptotically correct solution for the limit $t_f \gg 1/V$ relevant to the nFES [2, 6]. We will only consider the case where there is one channel in each electrode. As in the equilibrium case (cf. 35)

$$R = \exp(\lambda \log S(t)). \tag{59}$$

In this case the solution for $Y(t)$, valid for $t \gg V^{-1}$, is given for $t < 0$ or $t > t_f$ by

$$Y(t, \lambda) = \psi(t, \lambda), \tag{60}$$

while above and below the cut, $[0, t_f]$,

$$\begin{aligned} Y_+(t, \lambda) &= \begin{pmatrix} 1 & -\gamma(t,\lambda) \\ 0 & 1 \end{pmatrix} \psi_+(t, \lambda) \\ Y_-(t, \lambda) &= \begin{pmatrix} 1 & 0 \\ +\eta(t,\lambda) & 1 \end{pmatrix} \psi_-(t, \lambda). \end{aligned} \tag{61}$$

Here $\gamma(t, \lambda) = R_{12}/R_{11}$ and $\eta(t, \lambda) = R_{21}/R_{11}$. The functions $\psi_\pm(t, \lambda) = \psi(t \pm i0, \lambda)$, where $\psi(z, \lambda)$ is given by

$$\psi = \exp\left[(x_1\tau_0 + x_2\tau_3) \log \frac{z}{z - t_f}\right], \tag{62}$$

with

$$x_1(\lambda) = \frac{\log R_{11}/R_{22}^*}{4\pi i} \quad \text{and} \quad x_2(\lambda) = \frac{\log (R_{11}R_{22}^*)}{4\pi i}. \tag{63}$$

Here τ_3 is the third Pauli spin matrix and τ_0 is the identity matrix. The derivation of (61) follows that given in [2]. The idea, which was explained in detail in the context of inverse scattering problems in [4], is to solve for a function $W(z)$ which satisfies the same jump condition as $Y(z)$ but in a complex plane with additional cuts. For this problem, the additional cuts are parallel to the imaginary axis and run from the branch points at $z = 0$ and $z = t_f$ to infinity. The discontinuities in $W(z)$ across the vertical cuts scale as $e^{-|Vz|}$. If Y is approximated by W, the errors in $\log \chi_R$ defined in (13) are only $O(1/Vt_f)$, and can, in principle, be computed order by order in powers of $(Vt_f)^{-1}$.

The form for $\log \chi_R$ for $t_f \gg V^{-1}$ is found by inserting (61) into (26) and (13) and computing the integrals over t and λ as in the equilibrium case [6]:

$$\log \chi(t_f, V) = -i(E_0 - \Delta(V))t_f - \beta' \log(iVt_f) + D, \tag{64}$$

where $\Delta(V)$ is given by the non-equilibrium generalization of Fumi's theorem [19, 20]

$$\Delta(V) = \int_{-\infty}^{0} \frac{\mathrm{tr} \log(S^e(E))}{2\pi i} dE + \int_0^V \frac{\log(S_{11}^e(E))}{2\pi i} dE. \tag{65}$$

The constant β' is given by (cf. 53):

$$\beta' = \left(\frac{\log(S_{11}^e)}{2\pi i}\right)^2 + \left(\frac{\log(1/S_{22}^e)^*}{2\pi i}\right)^2. \tag{66}$$

The constant term D can be estimated by requiring that the form for $\log \chi$ (65) matches the equilibrium one at $t_f = V^{-1}$ (38) valid for $t_f \ll V^{-1}$. This constant gives the contribution from excitations with frequencies between V and ξ_0. This gives:

$$D = \beta \log \xi_0/V. \tag{67}$$

The result for $G(t_f)$ can be seen as an adaptation of the equilibrium result. The real phase shifts (given by

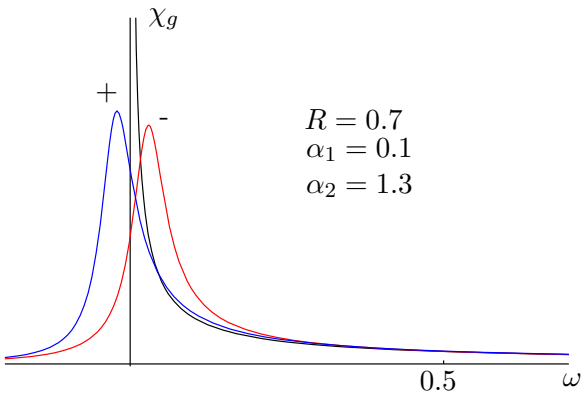

FIG. 3: Spectral function $\mathrm{Re}\chi_G(\omega)$ computed from (B10) with ω in units of the bias voltage V. The spectra depend on $S^e_{11} = \sqrt{R}e^{i2\alpha_1}$ and $S^e_{22} = \sqrt{R}e^{i2\alpha_2}$, where R is the reflection probability. The curve marked + (−) refers to the case in which electrode 1 (2) is at the higher chemical potential. Also shown is the corresponding equilibrium result calculated from (53) using $\xi_0 = V$ ([29]). In addition to the overall smoothing of the singularities, expected in a non-equilibrium system, there are two significant non-equilibrium features. Firstly, the maximum in the spectral weight is shifted away from its equilibrium value by an amount proportional to the applied voltage. The shift, $\mathrm{Re}(\Delta(V) - \Delta(0))$, which is given in the forward-biased case in (B4), depends on the polarity of the voltage. Secondly the form of the function changes on reversing the polarity of the device.

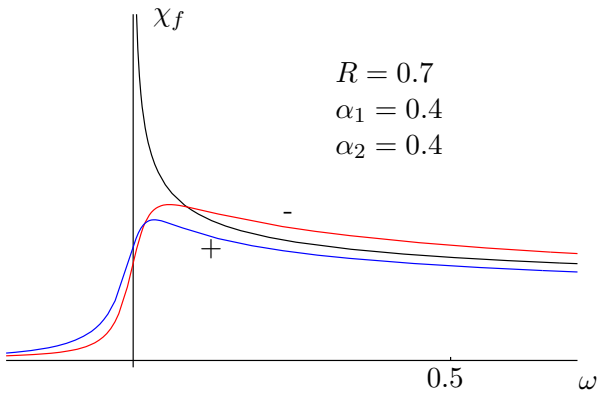

FIG. 4: Spectral function $\mathrm{Re}\chi_F(\omega)$ computed from (B10 with ω in units of the bias voltage V for the case $(W_1, W_2) \sim (0, 1)$ in (51). The curve marked + (−) refers to the case in which electrode 1 (2) is at the higher chemical potential. Also shown is the corresponding equilibrium result [29]. For these relatively small phase shifts the singularity seen in equilibrium disappears completely, although there is still a polarity dependence of the spectrum even though the scattering matrix is symmetric.

$-i$ times the logarithms of the eigenvalues of the scattering matrix S^e), which appear in the formulas (53), are replaced by complex phase shifts. In the forward bias case described by (53), these are $-i\log S^e_{11}$ and $-i\log(1/S^e_{22})^*$. The effect of the complex phase shifts is to smooth the singularity seen in equilibrium (this could be expected on quite general grounds) and to introduce a polarity dependence. This polarity dependence affects both the shape and the position of the spectrum and is evident in Figure 3 where we show $\chi_G(\omega)$ for a particular choice of S^e. The dependence of the spectrum, $\chi_G(\omega)$, on the polarity of the device, when operating out of equilibrium, is governed by the difference $\alpha_{12} \equiv \alpha_1 - \alpha_2$ (with $\alpha_{1,2}$ as defined in the figure caption). The difference in the overall position of the spectrum on changing the polarity is given by the difference in the second term on the right hand side of (65) and is proportional to α_{12}. This origin of this shift of the spectrum is the change in the nature of the scattering across the barrier from fully coherent in the equilibrium case to incoherent for times $t_f \gg V^{-1}$ in the non-equilibrium case. The shape of the spectrum reflects the decay of charge from its initial distribution (the equilibrium distribution for $S = 1$) to the steady-state distribution for $S = S^e$ [9, 22]. In the non-equilibrium case, this decay can occur differently depending on the polarity. If more charge is needed in the left-hand electrode to screen the potential characterized by S^e, than in the right-hand one ($\alpha_{12} > 0$), this charge can come from states within V of the Fermi energy of the right-hand electrode when the device is reverse-biased but not when it is forward-biased.

For the model device shown in Fig 2 the absorption spectrum is given by the Fourier transform (see Appendix B) of $G(t_f)$ rather than $F(t_f)$, as the transition in the barrier is presumed to be dipolar. However, the corresponding function $F(t_f)$ is also important. In [30, 31] Yuval and Anderson showed that the Kondo problem could be treated as an infinite sequence of spin flips or switching events, with the response of the conduction electrons to each switching event characterized by $F(t_f)$. Given the long-standing interest in non-equilibrium effects in the Kondo effect [32, 33], the correct non-equilibrium form for $F(t_f)$ would be the starting point for the study of the non-equilibrium Kondo effect using a generalization of the Yuval-Anderson mapping.

We must first write the function F and the open line function L in terms of the gauge-transformed basis:

$$L(t_f) \simeq W_i^* \sqrt{\nu_i} e^{-iLVt_f} \left[Y_-(t_f) \frac{1}{it_f} Y_+^{-1}(0) \right]_{ii'} W_{i'} \sqrt{\nu_{i'}}. \tag{68}$$

We can now insert the solution for Y_+ (with $\lambda = 1$) from (61) into (51). The result can be written:

$$L(t_f) \simeq \frac{1}{it_f} \mathbf{W}^* \begin{pmatrix} z_f (iVt_f)^{2x_+} & 0 \\ \alpha z_f (iVt_f)^{2x_+} & (iVt_f)^{2x_-} \end{pmatrix} \mathbf{W}, \tag{69}$$

where $x_\pm = x_1(1) \pm x_2(1)$ with $x_i(\lambda)$ defined in (63), $\alpha = S^e_{21}/S^e_{11}$ and $z_f = e^{-iVt_f}$ [34]. The absence of a contribution proportional to $W_1^* W_2 z_f$ is to be expected. This would involve a contribution to the open-line function from an electron initially placed in the right-hand electrode exciting the Fermi gas in the left electrode.

Since we are assuming that the tunneling through the barrier by the electron is a slow process on the scale of $1/V$, this does not lead to a singular contribution to F. (There is still a contribution to F proportional to $W_1^* W_2 z_f$ from the direct scattering term $CG(t_f)$ in 48.)

The effect of the open line contribution on $F(t_f) = G(t_f)L(t_f)$ is the natural generalization of the equilibrium result that one might expect given the results for $G(t_f)$. The corresponding spectral functions is shown in Figure 4 for a particular choice of S^e. For simplicity we only look at the case where the electron is added and removed from the same ($i = 2$) electrode, *ie* $\mathbf{W} \sim (0, 1)$. The dependence of the spectrum on the polarity of the device, is present even in the case where S^e is symmetric. When a particle is added to an electrode, the response of the system will be depend on whether the electrode is at the higher or lower chemical potential. The form of the spectrum can also differ substantially from what happens in equilibrium. For the case $\alpha_1 = \alpha_2 = 0.4$ and $R = 0.7$ shown in Figure 4, there is no real peak left over from the equilibrium result. This is because the phase shifts δ_ζ, corresponding to the eigen modes of S^e, are small, and hence the exponents in (53) for the equilibrium function $\chi_F(\omega) \sim \omega^{-(\delta_\zeta - \pi)^2/\pi^2}$ are also small. The corresponding singularity is weak and easily smoothed out by the finite lifetimes of states close to the Fermi energy in the non-equilibrium case. This smoothing is enhanced because one of the phase shifts, δ_ζ, is always larger than α_1 and α_2. This larger exponent gives the dominant singularity in equilibrium, but is then effectively replaced by α_1 out of equilibrium.

IV. CHARGE TRANSFER: (CSAC)

The existence of Coherent States of Alternating Current (CSAC) was predicted in [1]. These consist of a sequence of pulses which propagate through a contact. When the bias across the contact is described by a class of periodic (with period Ω) rational functions of the variable $z = e^{i\Omega t}$, then the shot noise is minimised and the noise distribution does not depend on the separation of the pulses. This result is still not well understood, nor is it possible, using the original derivation, to establish how robust these states are against deviations from zero temperature or from the ideal pulse shape.

Recent rapid experimental progress in the application of micro-wave radiation at low temperatures suggests that the experimental test of the existence of the CSAC is just about possible. Several experimental groups are pushing the technology in this direction [35, 36, 37], and it should only be a matter of time before experimental data becomes available. However, interpretation of these future experiments will not be easy using the analytical method used in [1] as this depends crucially on the particular shape of the pulses. There are no predictions about what happens when the shape of the pulses deviates slightly from the required one (something unavoidable in any real experiment), nor is the effect of non-zero temperature known.

Here we show that the results of [1] for the CSAC are easy to derive using the RH approach. When the pulses are periodic as in the case of the CSAC, the RH problem simplifies. It requires solving for functions which are analytic in two **disconnected** regions (inside and outside the unit circle) with the jump function specifying the discontinuity across the boundary between them. We show that the particular case of the CSAC corresponds to an RH problem which can be solved exactly using combinations of meromorphic functions in the plane—one of which is analytic inside and one outside the unit circle.

The model device considered in [1, 2] consists of a tunnel junction driven by a bias voltage $V(t)$ which is periodic in time with period $T = 2\pi/\Omega$. It is equivalent to the device shown in Fig 2. We are interested in the change in physical quantities over one cycle of the pump in the limit $t_f \gg T$. In this limit, effects induced by the switching on and off of the periodic potential at $t = 0$ and $t = t_f$ are irrelevant. The scattering matrix S is at some constant value S^e between 0 and t_f. Applying the time-dependent gauge transformation (58) leads to S becoming a periodic function of time, so that it no longer commutes with itself at different times.

The distribution function for any single-particle observable measured in this periodically pumped Fermi system will involve the solution of a non-commuting RH problem. In particular, the characteristic function or generating function for moments of the distribution of the net transfer of charge from electrode 1 (left electrode) to the electrode 2 (right electrode), $\chi(\lambda)$, is given by (4) with the operator R given by

$$\hat{R}(\lambda) = \hat{U}^\dagger(t_f) e^{-i\lambda \hat{Q}_1} \hat{U}(t_f) e^{i\lambda \hat{Q}_1}. \tag{70}$$

Here

$$\hat{Q}_1 = \sum_\epsilon \hat{a}_\epsilon^\dagger L \hat{a}_\epsilon. \tag{71}$$

For states close to the Fermi surface ($E = \epsilon + \epsilon' \simeq 0$), the matrix R in the time-representation can be written

$$R(t, \lambda) = S^{-1}(t) e^{-i\lambda L} S(t) e^{i\lambda L}, \tag{72}$$

so that the characteristic function will be given by (12, 13, 25)

$$\log \chi(\lambda) = \mathrm{Tr}\left(\ln\left(1 - f + fR\right)\right). \tag{73}$$

If the inverse of the solution $Y^+(t)$ to the RH problem (22, 23) with R given by (72), is analytic in the upper half-plane, we can write the characteristic function as

$$\log \chi(\lambda) = \int_0^\lambda d\lambda' \int dt\, \mathrm{tr}\left\{ Y_+ f Y_+^{-1} R^{-1} \frac{dR}{d\lambda'} \right\} \tag{74}$$

Using (72) and (21), and computing explicitly the derivative with respect to λ, we obtain [2, 38]

$$\log\chi(\lambda)=\int_0^\lambda\frac{d\lambda'}{2\pi}\int dt\,\mathrm{tr}\left\{\frac{d(Se^{i\lambda' L}Y_+)}{dt}(Se^{i\lambda' L}Y_+)^{-1}\right\}. \tag{75}$$

(If the eigenvalues of Y_+ have zeros in the upper half-plane, there are additional contributions to the right hand side of the corresponding relations to (24) for Y_+^{-1} from its poles. In this case, $(1-f+fR)^{-1}$ is no longer given by (25) but can be found using methods described in Chapter 6 of [16].)

In the case of the periodically driven pump, the scattering matrix (after applying the gauge transformation) is periodic $S(t)=S(t+T)$. If we change variables to $z=e^{2\pi i t/T}$, we need to find a function $Y_+(z)$, which is analytic for $|z|<1$ and Y_- which is analytic for $|z|>1$ and $Y_-\to 1$ when $|z|\to\infty$. On the unit circle $|z|=1$,

$$Y_-Y_+^{-1}=S^{-1}(z)e^{-i\lambda L}S(z)e^{i\lambda L}. \tag{76}$$

The characteristic function for charge transmitted during one cycle of the periodic pump in the limit $t_f/T\gg 1$ is given by

$$\begin{aligned}\log\chi &= \int_0^\lambda\frac{d\lambda}{2\pi}\oint_{|z|=1}dz \\ &\quad \mathrm{tr}\left\{\frac{d\left(Se^{i\lambda L}Y_+\right)}{dz}\left(Se^{i\lambda L}Y_+\right)^{-1}L\right\}\end{aligned} \tag{77}$$

CSAC's were reported in [1] for the case when the phase factor in (58) can be written as a rational function, $l(z)$, of the variable $z=e^{2\pi i t/T}$:

$$\begin{aligned}e^{+iL\int_0^t V(\tau)d\tau} &= l(z)\\ l(z) &= \prod_{i=1}^N\frac{z-a_i}{1-a_i^* z},\end{aligned} \tag{78}$$

where either all $|a_i|>0$ or $|a_i|<0$. We can choose $|a_i|>0$ without loss of generality as $z\mapsto 1/z$ simply reverses the polarity of the device. In this case, we decompose $R(t,\lambda)$ (see [2]) as follows

$$R=\begin{pmatrix}1 & 0\\ \frac{\alpha}{l(z)} & 1\end{pmatrix}\begin{pmatrix}a & 0\\ 0 & \frac{1}{a}\end{pmatrix}\begin{pmatrix}1 & \beta l(z)\\ 0 & 1\end{pmatrix}, \tag{79}$$

where

$$\begin{aligned}a &= |S_{12}|^2e^{i\lambda}+|S_{11}|^2\\ \alpha &= -\frac{S_{21}^e\left(S_{22}^e\right)^*\left(1-e^{i\lambda}\right)}{a}\\ \beta &= -\frac{S_{12}^e\left(S_{11}^e\right)^*\left(1-e^{i\lambda}\right)}{a}.\end{aligned} \tag{80}$$

The solution to the RH problem

$$Y_-Y_+^{-1}=\begin{pmatrix}1 & 0\\ \frac{\alpha}{l(z)} & 1\end{pmatrix}\begin{pmatrix}a & 0\\ 0 & \frac{1}{a}\end{pmatrix}\begin{pmatrix}1 & \beta l(z)\\ 0 & 1\end{pmatrix}, \tag{81}$$

is clearly

$$\begin{aligned}Y_- &= \begin{pmatrix}1 & 0\\ \frac{\alpha}{l(z)} & 1\end{pmatrix}\\ Y_+ &= \begin{pmatrix}1 & -\beta l(z)\\ 0 & 1\end{pmatrix}\begin{pmatrix}\frac{1}{a} & 0\\ 0 & a\end{pmatrix}.\end{aligned}$$

Inserting this into (77) gives the result reported in [1]:

$$\ln\chi(\lambda)=N\frac{t_f V}{2\pi}\ln a \tag{82}$$

with a given by (80).

The surprising feature of the result (82) is that it implies that the second moment of the shot noise $\ll n^2\gg$ achieves the absolute minimum for given charge transfer $<n>$ [1], which is the value obtained in the constant bias case $a_i\to\infty$ for all i. The feature of the phase factor $l(z)$ which leads to the RH problem being so easy to solve is that all its poles (zeros) are either inside or outside the unit circle $|z|=1$, which means that the decomposition of R in (79) automatically solves the RH problem. In the case of an arbitrary rational function for $l(z)$ this is not the case as there can be points at which $\det|Y_+(z)|$ vanishes inside the unit circle. The corresponding formulae to (24) Y_+^{-1} pick up additional terms on the right hand side nad $(1-f+fR)^{-1}$ is not given by (25), although, in principle, it can still be found given the solution to the RH problem $Y(z)$.

V. CONCLUSIONS AND OUTLOOK

The RH approach is a general method for computing the response of a Fermi gas to a localized time-dependent perturbation. There are two key steps to the method. First, provided the condition (5) is met, the method works with the scattering matrix defined on the instantaneous value of the potential rather than with the Hamiltonian. This has the attractive feature of working directly with the physical quantities determining the long time response of the system to a perturbation, namely scattering amplitudes for particles close to the Fermi surface. The condition (5) is essentially the requirement that the perturbation varies more slowly than the delay time for a particle traversing the region in which the perturbation acts. The second key step is to relate the response of the Fermi gas to the solution of a non-commuting RH problem (22, 23). The RH problem corresponding to any given experimental situation is usually easy to set up. Its solution and the interpretation of the results is a more delicate task which needs to be repeated for each new physical situation. While there is no analytical solution of the general non-Abelian RH problem, there is a powerful technique for finding asymptotic solutions valid for frequencies much smaller than those present in the jump function [4].

Here we have emphasized the generality of the approach and applied it to two existing problems—the

Fermi Edge Singularity and the shot noise in a periodically pumped tunnel junction. The calculations in the two cases are very similar. In the case of the FES we have rederived all the known results for the equilibrium case emphasizing, in particular, how the method is no more complicated in the case of the non-separable potential than in the separable case. Our derivation for the non-separable case is, we believe, the first non-perturbative solution of this problem. For the non-equilibrium device shown in Fig 2, we have explained how the results for the core-hole Green's function of [6] were obtained and given the corresponding results for the open-line function $L(t_f)$ (69). For the case of the CSAC's, we have shown that the particular form of the periodically varying bias with the phase factor $l(e^{2\pi it/T})$ given by (78) corresponds to a case in which the RH problem can be solved exactly.

It is possible within the RH approach to handle corrections to the asymptotic solution to the non-communting RH problem we have been using in order to allow us to compute the response of systems in the intermediate regime (where one is interested in the response at frequencies comparable to those introduced by the perturbation). The RH problem lends itself naturally to a type of perturbative analysis. The corrections to the approximate solution valid for long times, (60) and (61), can be described by multiplying the approximate solution by a function which is analytic except across the additional vertical cuts introduced to simplify the original problem. This function can be specified by a Cauchy integral around the cut. Preliminary work in this direction has been attempted in [39].

Finally, the RH method should generalize to non-zero temperatures. As was observed in [26], the singular integral equation appearing at finite temperatures in a related problem can be solved analytically. Also, the analytic treatment of the finite-temperature Fermi-edge singularity in [17, 40] again suggests that the RH approach will generalize successfully to finite temperatures.

APPENDIX A: BOUND STATES

If the perturbing potential generates a bound state(s), then (6) is no longer correct. In the case where the potential (and hence $S(t)$) simply switches between its unperturbed value and a new but time-independent value at $t = 0$ and back again at $t = t_f$, we can correct σ by including the effect of the bound state explicitly. The treatment follows closely that of [11], although only the case of a separable potential was treated there. We write

$$\sigma = \tilde{\sigma} + e^{iH_0 t}|b\rangle e^{-iE_b t}\langle b|. \tag{A1}$$

Here $|b\rangle$ is the bound state wavefunction, while $\tilde{\sigma}$ describes the scattering of the states within the continuum, and is given by the Fourier transform of the scattering matrix $S(t)$ (6) as before. (H_0 is the matrix of $\hat{H}_0$ taken between single-particle basis states.)

For the case of the function $G(t_f) = \det|1 - f + f\sigma|$ (see 31) we have

$$\begin{aligned} G(t_f) &= \tilde{G}(t_f)\det|1 + A|b\rangle\langle b||\\ &= \tilde{G}(t_f)\,(1 + A_B)\,, \end{aligned} \tag{A2}$$

where $A_B = \langle b|A|b\rangle$ with

$$A = (1 - f + f\tilde{\sigma})^{-1} f e^{iH_0 t} e^{-iE_b t}, \tag{A3}$$

and where $\tilde{G}(t_f) = \det|1 - f + f\tilde{\sigma}|$. We write the bound state as an expansion over the basis vectors

$$|b\rangle = \sum_{\epsilon} \mathbf{u}_\epsilon \cdot \hat{\mathbf{a}}^\dagger_\epsilon|\rangle. \tag{A4}$$

For long times t_f the response is dominated by states within $1/t_f$ of the Fermi energy and it is a reasonable approximation to neglect the energy dependence of the coefficients $\mathbf{u}_\epsilon$. After switching to the time-representation, and using (49) with $\tilde{\sigma}$ in place of σ, we obtain (ν_l is the density of states in channel l)

$$\begin{aligned} \langle b|A|b\rangle &= e^{-iE_b t_f}\sum_{ll'}\int d\epsilon d\epsilon' \int dt_1 dt_2 u^*_{l'} u_l \times \\ &\sqrt{\nu_l \nu'_l} e^{i\epsilon' t_1}\left[Y_+ f Y_-^{-1}\right]_{lt_1, l't_2} e^{i\epsilon(t_f - t_2)}. \end{aligned} \tag{A5}$$

Integrating over energies and times gives

$$A_B \sim e^{-iE_b t_f} u_l \sqrt{\nu_l}\left[Y_+(0)\frac{1}{-it_f}Y_-^{-1}(t_f)\right]_{ll'} u^*_{l'}\sqrt{\nu_{l'}} \tag{A6}$$

Provided $S(t)$ commutes with itself at all times between 0 and t_f, Y is given by (37). For the single-channel case with $S = e^{2i\tilde{\delta}}$, we obtain

$$\langle b|A|b\rangle \sim \frac{\nu}{it_f}\frac{1}{(i\xi_0 t_f)^{2\tilde{\delta}/\pi}}. \tag{A7}$$

Here we introduce the quantity $\tilde{\delta}$ which is the phase shift modulo π and takes values on the interval $[-\pi/2, \pi/2]$. Normally the phase shift δ is defined with a jump of π at a bound state thereby ensuring compliance with the Friedel sum rule [20]. However, when writing the scattering matrix as in (A1), the contribution from the bound state to the scattering matrix is explicitly included in the second term on the right hand side and is not in the scattering matrix S. At the bottom of the band, the value of the phase shift which enters the threshold shift is clearly $\tilde{\delta}$ as emphasized in [11].

Although the calculation is longer, the function $F(t_f)$ can be obtained in a similar manner by replacing σ in (46) by the form (A1). One needs only to keep track of terms up to first order in $e^{-iE_b t_f}$. (Higher order terms must give zero as they correspond to double or higher occupancy of the bound state. They can be seen to make no contribution by subsituting the formula (A1) in (42).) As for the case of the function $G(t_f)$ considered above,

we neglect the energy dependence of $\mathbf{W}_\epsilon$ and $\mathbf{u}_\epsilon$ (see 29 and A4). We define

$$C_b = (\mathbf{W}^* \cdot \mathbf{u})(\mathbf{u}^* \cdot \mathbf{W}) e^{-iE_b t_f} \quad \text{(A8)}$$

$$\tilde{O} = (1 - f + f\tilde{\sigma})^{-1} f. \quad \text{(A9)}$$

Here $\tilde{O}$ is just the scattering state contribution to O (see 47):

$$O \simeq \tilde{O} - \tilde{O} e^{iH_0 t_f} |b\rangle e^{-iE_b t_f} \langle b| \tilde{O}. \quad \text{(A10)}$$

We obtain (from 48)

$$\begin{aligned} F(t_f) &= G(t_f) \left[C - \langle g|O|h\rangle \right] \\ &\simeq \tilde{G}(t_f) C_b - \tilde{G}(t_f) \left[1 + A_B \right] \langle g|O|h\rangle. \end{aligned} \quad \text{(A11)}$$

Retaining the dominant terms and ignoring the possibility that there is an unexpected cancellation between terms proportional to $e^{-iE_b t_f}$,

$$F(t_f) = \tilde{G}(t_f) \tilde{L}(t_f) + a C_b \tilde{G}(t_f) \quad \text{(A12)}$$

where $a \sim 1$ is some constant and $\tilde{L}(t_f)$ is the scattering state contribution to the open-line function. For the single channel case with $S = e^{i2\delta}$, we again assume that the exponent in $\tilde{G}(t_f)$ is $\tilde{\delta} = \delta - \pi$ and obtain $F(t_f) \sim F_b(t_f) + F_0(t_f)$ with

$$F_b(t_f) \sim e^{-iE_b t_f} \frac{1}{(i\xi_0 t_f)^{(\tilde{\delta}/\pi)^2}}, \quad F_0(t_f) \sim \frac{1}{(i\xi_0 t_f)^{(\tilde{\delta}/\pi - 1)^2}}. \quad \text{(A13)}$$

APPENDIX B: COMPUTING SPECTRAL FUNCTIONS

Given $G(t_f)$ or $F(t_f)$ we would like to compute the corresponding spectral functions given by a Fourier integral over t_f. Assume that scattering matrix, S^e, has diagonal elements $\sqrt{R} e^{i2\alpha_{1,2}}$. Using the complex cutoff ζV (normally $\zeta = i$), we have from (64)

$$\log G(t_f, V) = -i(E_0 - \Delta(V)) t_f - \beta_G \log(\zeta V t_f) + D. \quad \text{(B1)}$$

The exponent $\beta_G = x_+^2 + x_-^2$, where

$$x_+ = \frac{\log S^e_{11}}{2\pi i} = \frac{\alpha_1}{\pi} - i\frac{\log R}{4\pi} \quad \text{(B2)}$$

and

$$x_- = \frac{\log(1/S^{e*}_{22})}{2\pi i} = \frac{\alpha_2}{\pi} + i\frac{\log R}{4\pi}. \quad \text{(B3)}$$

The modified threshold shift is given by (65)

$$\Delta(V) = \left(\Delta(0) + V \frac{\alpha_1 - (\log S^e)_{11}}{\pi} \right) - iV \left(\frac{\log R}{4\pi} \right) \quad \text{(B4)}$$

The real part of $\Delta(V)$ fixes the threshold. We will absorb this into the definition of frequency when computing Fourier transforms.

We write $\beta_G = \beta_{G1} + i\beta_{G2}$ with

$$\beta_{G1} = \left(\frac{\alpha_1}{\pi}\right)^2 + \left(\frac{\alpha_2}{\pi}\right)^2 - \frac{1}{2}\left(\frac{\log R}{2\pi}\right)^2 \quad \text{(B5)}$$

and

$$\beta_{G2} = -\frac{(\alpha_1 - \alpha_2)}{\pi} \frac{\log R}{2\pi}. \quad \text{(B6)}$$

For the function $F(t_f)$, the exponent becomes $\beta_F = (x_- - 1)^2 + x_+^2$ or $\beta_F = x_-^2 + (x_+ - 1)^2$ depending on whether the electron is added to the electrode with lower or higher chemical potential. This gives $\beta_F = \beta_G - 2x_\pm + 1$ and

$$\beta_{F1} = \beta_{G1} - \frac{2\alpha_{1,2}}{\pi} + 1 \quad \text{and} \quad \beta_{F2} = \beta_{G2} \pm \frac{\log R}{2\pi}. \quad \text{(B7)}$$

Introducing

$$\omega_2 = -\log R / 4\pi, \quad \text{(B8)}$$

the spectral functions of G or F are proportional to the real part of the Fourier integral, $\chi_{F,G}(\epsilon)$, where:

$$\chi(\epsilon) = \int_0^\infty dt_f \, (\zeta V t_f)^{-\beta} \, e^{(i\epsilon - \omega_2 V) t_f}, \quad \text{(B9)}$$

with $\beta = \beta_F$ for χ_F and β_G for χ_G. Here the lower limit of the integral is taken to be 0, which is only valid when $\beta_1 < 1$. When $\beta_1 > 1$, contributions from the lower limit of the integral dominate and the response is dominated by high frequency contributions which are not changed from the equilibrium case. These are not described by the formula (64) and depend on details relating to the band edge. If the phase shifts $x_\pm$ are small, which can be the case for the spectral function of F (or $\tilde{G}A_B$ in the presence of a bound state, see A2 and A7), then β_1 will be close to 1. In this case $\chi(\epsilon)$ given in (B9) contains a significant contribution from times $t_f < 1/V$ for which our asymptotic solution for $F(t_f)$ (or $\tilde{G}A_B$) is incorrect. We can correct for this by noting that when $1 - \beta \ll 1$ the contribution from times with $Vt_f < 1$ gives just a constant offset which can be subtracted from χ. To see this, we expand the exponential term $e^{(i\epsilon - \omega_2 V)t}$ in the integrand and integrate term by term from $Vt_f = 0$ to $Vt_f = 1$. The first term in the expansion is independent of ϵ and much larger than subsequent terms provided $(\epsilon/V) \ll 1/|1 - \beta|$. In practice we subtract from the real part of χ its value at $\omega \simeq -V$. (When $1 - \beta$ is not small the contribution from the times $Vt_f < 1$ to the real part of χ is negligible anyway.)

Eq (B9) is in the form of a standard integral and (see 8.312.2 in [41]) is given by:

$$\chi(\omega_1) = (i\zeta)^{-\beta} \frac{i}{V} \left(\frac{1}{\omega_1 + i\omega_2} \right)^{1-\beta} \Gamma(1 - \beta). \quad \text{(B10)}$$

If we define $\Omega = |\omega_1 + i\omega_2|e^{i\phi_\Omega}$ and write:

$$\Gamma(1-\beta) = |\Gamma(1-\beta)|e^{i\phi_\Gamma} \quad \text{and} \quad i\zeta = e^{i\phi_\zeta}, \qquad \text{(B11)}$$

then

$$\chi(\omega_1) = e^{-i\beta_1\phi_\zeta + \beta_2\phi_\zeta}\frac{i}{V}\frac{e^{i\beta_2\log\Omega}}{\Omega^{1-\beta_1}} \times e^{i(\beta_1-1)\phi_\Omega - \beta_2\phi_\Omega}|\Gamma(1-\beta)|e^{i\phi_\Gamma}. \qquad \text{(B12)}$$

The real part of $\chi(\omega_1)$ can then be written [29]:

$$\mathrm{Re}\chi(\omega_1) = \frac{|\Gamma(1-\beta)|}{V}\frac{1}{\Omega^{1-\beta_1}}e^{-\beta_2(\phi_\Omega-\phi_\zeta)} \sin\left[\beta_1(\phi_\zeta-\phi_\Omega) + (\phi_\Omega-\phi_\Gamma) - \beta_2\log\Omega\right]. \qquad \text{(B13)}$$

For both functions F and G, the cutoff parameter $\zeta = i$, so $\phi_\zeta = \pi$.

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[34] The factor $z_f = e^{-iVt_f}$ in (69) measures the energy to move from one chemical potential to the other, but its effect on thresholds will depend on how the bias voltage is applied. Here we have assumed that $l = 2$ electrode is earthed and the potential on the $l = 1$ electrode is varied.

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$$\int dt Y_+ f Y_+^{-1} L = \int dt f Y_+ Y_+^{-1} L. \qquad \text{(B14)}$$

This follows from (23).

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